DESIGN CHALLENGES FOR ELECTRONIC MULTIBEAM STEERING OF BROADBAND HIGH THROUGHPUT SIGNALS FOR SATELLITE COMMUNICATION

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Low Earth Orbit (LEO) has emerged as the latest frontier for commercial endeavors, with an escalating number of satellites being deployed for various applications. No matter what the service provided, continuous tracking of these satellites is essential for their operation and interaction with ground-based systems. However, the sheer growth in their numbers is rendering dedicated reflector-based antenna systems increasingly inefficient. Innovations are underway for both user terminals and tracking stations, also known as Gateways. This presentation delves into the design considerations and challenges associated with this evolving technology.

INTRODUCTION

CelestiaUK is part of the Celestia Technologies Group -CTG, which is a pan European group of companies specialising in the development of cutting-edge technologies for Ground based Satellite communication systems.

We are experts in Antenna, Radio Frequency and Digital Signal Processing technologies. Our R&D portfolio includes a variety of different subject matters, including Beamforming, Satcom Modems, RFIC/MMIC design, Digital Signal Processing, Real-time Software and Time Synchronization algorithms.

CelestiaUK is based in Edinburgh. Our Team has embarked on an ambitious programme to design an innovative Phased-Array Antenna solution for a High Throughput Satellite system, as part of a large, multi-phase programme funded by the European Space Agency.

This technology is based on the electronic beam steering concept. Its purpose and the design challenges associated to its implementation are described below.

Beam Steering

The need for beam steering is a consequence of both Antenna Directivity and moving targets.

Antenna directivity is required for energy efficiency and to limit interference. Many natural sources, like the stars, are isotropic radiators and this is why we can see them in the night sky, when the interfering light from the Sun is not present. If they were directive sources, we would not see them, unless they were pointing in our direction.

Energy spreads equally in all directions in isotropic mediums like the space and the atmosphere, and a convenient manner to express power flux, ϕ (power per unit area) is to assume that the power radiated by an antenna is spread over the surface of a sphere of radius r, equal to the distance to the radiating source. To account for the fact that the radiating source does not equally send its energy in all directions, and thus the radiated power does not spread uniformly over the surface of the sphere, a term called Directivity **D**, is applied which is a function of direction, defined by its azimuth φ, and polar θ in a spherical coordinate system (Fig *1*).

$$\Phi = \frac{P_{rad}}{4\pi r^2} \cdot D$$

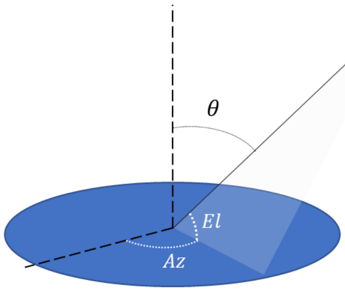

Fig 1. Beam directions in azimuth (Az), elevation (El) and scan angle (θ)

The directivity can take any value from zero to infinity (in theory), but it always fulfils the condition.

$$\int_{4\pi} D(\theta, \varphi) \cdot d\Omega = 1$$

Implying that the higher the directivity in a specific direction, the lower it is in all other directions (narrower beam shape). Consequently

$$\int_{4\pi} \phi \cdot d\Omega = P_{rad}$$

at any given distance **r**, in a lossless medium.

This implies that the total power flux from the antenna is produced by its radiated power.

In plane wave reception a certain amount of the power flux present at the antenna location is collected and converted into electrical power. The effective collecting area of an antenna is defined as[1]

$$A_e = \frac{\lambda^2}{4\pi} \cdot D$$

Implying that the higher the directivity, the greater the collecting power of the antenna in a specific direction. It is expressed as

$$P_{collected} = \frac{\lambda^2}{4\pi} \cdot D \cdot \phi$$

[1] This relation is due to Dicke (1946) and represents the collecting area of a lossless isotropic antenna. It results from integrating the incoming power flux in all directions, contributing to the total electrical power collected at a single polarization, as the antenna is a polarization sensitive device.
In thermal equilibrium inside a closed cavity, the power flux is expressed by the black body radiation (Planck, 1900) $\Phi = \frac{2kT}{\lambda^2} \cdot \frac{h\nu/kT}{e^{h\nu/KT}-1}$, and the electrical power at the antenna port is given by the power spectral density of the thermal noise (Nyquist, 1928) $\eta = kT \cdot \frac{h\nu/kT}{e^{h\nu/KT}-1}$, in its general form to account for quantum effects occurring at very high frequencies or very low temperatures near absolute zero. Statistically half of the total power flux contains the radiated thermal energy at a single polarisation. Energy conservation leads to the expression $\frac{1}{2}\int_{4\pi} \Phi \cdot A_{eff} \cdot d\Omega = \eta$, from where the result $4\pi \cdot A_{eff} = \lambda^2$ is obtained.

We are referring to *radiated* and *collected* powers because they are directly related to the Directivity of an antenna. Antenna Gain is preferably used in link budgets, and it is related to transmitted and received power. Thus, the antenna Gain accounts for internal losses and deals with the power levels at the electrical interface where the RF signal is used in the front-end chain. But when evaluating the performance of active phased arrays, it is more convenient to use the Directivity, as it accounts for the concentrating power of the antenna without considering the signal gain or loss provided by other components in the Transmitter or Receiver RF chains.

The directivity of an antenna is usually fixed at any given frequency. This is true for all antennas of fixed beam that perform their tracking by mechanically changing their orientation, thus pointing the beam in the required direction.

In *Electronically Steerable Antennas*, or ESA, the beam shape or directivity is dependent on the frequency and beam direction, which can be controlled by electrical (non-mechanical) means - *electronic steering*.

The beam direction is defined in a spherical coordinate system by its azimuth φ, and scan angle θ, where $\theta = 0$ corresponds to the boresight direction (non-dependent of φ) along the axis orthogonal to the plane of a flat panel array. This direction provides the highest beam directivity, which decreases at higher scan angles due to a phenomenon known as scan loss, related to a reduction of the effective radiating aperture. The term beam elevation is frequently used, and it is the complementary of the scan angle θ.

The radiating aperture is closely related to the geometry of a flat panel array or the circular shape of a dish antenna, where the plane wavefront is formed.

In dish antennas (Fig *2*) a plane wavefront is formed by the coherent contributions of all rays emitted from the antenna feed - placed at the dish's focus, as they are reflected by the dish in the direction of its main axis. The geometry of the paraboloid guarantees that all reflected paths have the same length and thus they all experience the same electrical delay, providing the same phase at the plane of the wavefront.

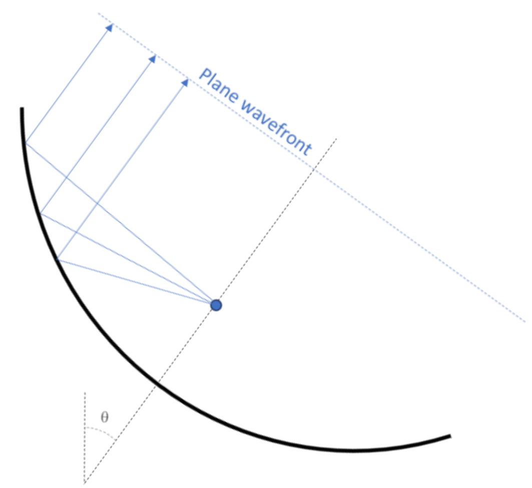

Fig 2. Mechanical steering by a fixed beam antenna

The same principle is used by ESAs (Fig *3*) to conform outgoing plane waves travelling in different directions by applying the corresponding electrical delays to the signals radiated by its individual elements, called *radiating elements* or REs.

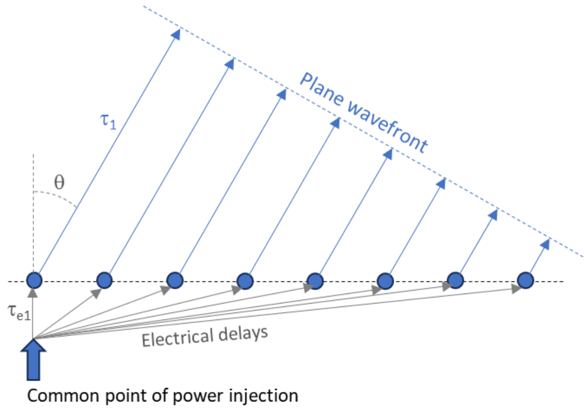

Fig 3. Electronic beam steering concept

Array dimensioning

The internal electrical delays τ_{ek} in Fig *3* are adjusted to fulfil the condition of equal total delay $\tau_{ek} + \tau_k$, between the common point of power injection and the reference plane wavefront, for any radiating element k. This is achieved by providing an incremental delay $\Delta\tau = d \cdot sin(\theta)$ between consecutive REs, with d being the separation distance. In a general case d may not be the same for all elements, but the working principle of the phased array is the same; All paths have the same electrical length and contribute coherently to the formation of a plane wavefront in the scan direction θ. In the same manner the azimuth direction φ can also be adjusted. It is the relative electrical delay of the signal applied to consecutive radiating elements which determines the direction of beam formation. The reception case is reciprocal.

The separation between radiating elements determines the maximum achievable scan angle while sidelobes and grating lobes are below the limit imposed by regulation masks. Fig 4 shows the maximum permitted electrical length between radiating elements as a function of the target maximum scan angle for the aperture. As it is presented, the electrical distance to meet 70 deg scanning range is $0.5 \cdot \lambda$.

This result is obtained by adding all the contributions from the radiating elements in Fig *3* and tuning the element-to-element distance to keep a secondary lobe rejection of at least 20 dB with respect to the main lobe amplitude. A derivation is shown in Appendix I, and Fig *5*-Fig *6* plot the results obtained. Note the decrease in beam amplitude as the beam is steered at higher scan angles. It is known as *scan loss* and is accompanied by an increase in beamwidth, implying a reduction in Directivity.

The aperture area (the size of the array) imposes its boresight Directivity, as the beam shape is determined by the extent of the array in the x and y directions (assuming the boresight beam pointing in the z-direction). Antenna apertures are dimensioned to guarantee a value of beam directivity within the required range of scan and azimuth angles (as defined in Fig *1*)

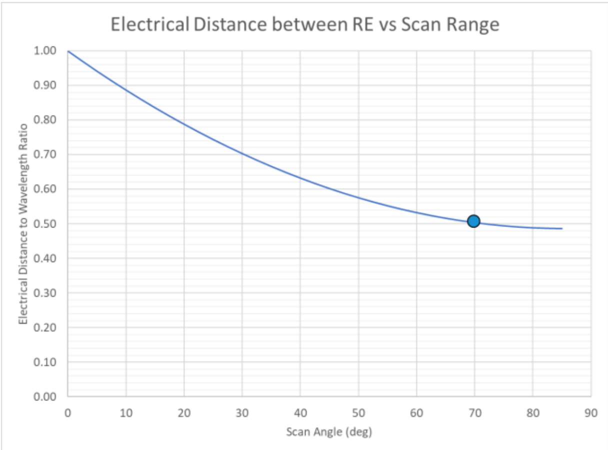

Fig 4. Electrical length required between radiating elements.

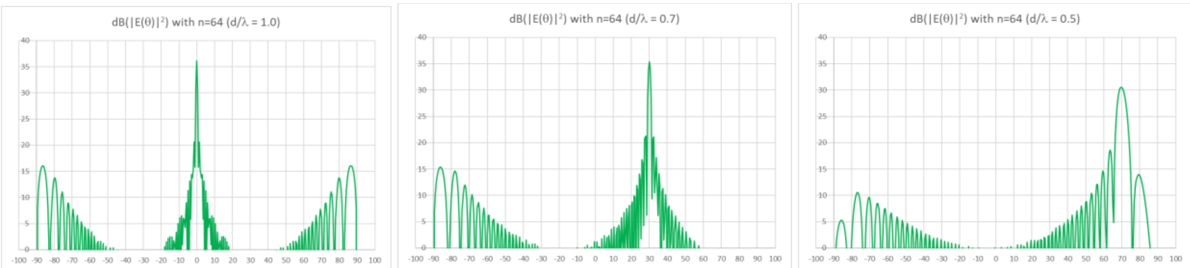

Fig 5. Element spacing requirements at different steer angles.

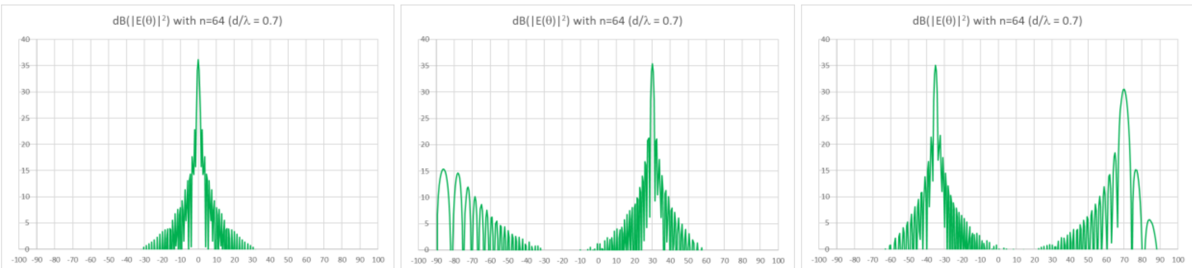

Fig 6. Beam produced at different steer angles with a fixed Element spacing (note the high grating lobe at around -35deg. appearing at the 70deg. high scan angle)

The electrical effective area of an aperture is proportional to its geometrical surface and reduces with the angle of view of the front wave as its direction of arrival departs from the boresight direction.

$$A_{eff}(\theta) = \eta \cdot S_{geom} \cdot Cos(\theta)^p$$

η is the aperture efficiency, and the exponent p models the *scan loses* and is found in practice to be around 1.2-1.4, although higher values can be measured in small apertures.
Boundary effects limit the validity of the simple analysis presented here when small apertures are considered. The larger the aperture size the more realistic is the behaviour of the radiating patterns and the scan losses.

The worst-case effective area corresponds to the maximum scan angle θ_{max}. This value is used to size the geometrical dimensions of the panel (S_{geom}) to achieve the required directivity. The number of radiating elements in the aperture is computed as:

$$nr.RE \geq \frac{S_{geom}}{\lambda^2 \cdot el_x \cdot el_y}$$

where $nr.RE$ is an integer number, el_x, el_y are respectively the electrical lengths in the x and y directions[2], and λ is the wavelength of the signal. The directions x and y are aligned with two orthogonal dimensions which, in the case of non-circular apertures, may be used to define different scan angle requirements in azimuth and elevation.

Active phased arrays are not simply antennas but complete transmitting and receiving systems of their own. Two important parameters determining a link budget are EIRP, determining the amount of radiated power in one specific direction, and G/T which determines the system's detection threshold for a particular type of signal or modulation scheme. They are both proportional to the antenna Gain, which is itself proportional to the Directivity and accounts for feeder losses.

There are two key elements in a phased array: the Radiating Elements, and the beamforming network.

Due to the requirement to keep the separation between radiating elements below one wavelength (and typically less than a half-wavelength), integrability is critical in high frequency applications like the K/Ka-band systems developed by CelestiaUK, and so the radiating elements are based on small form patch antenna structures that can easily be repeated in two spatial directions, forming a planar grid. The beamforming network is principally composed of active ICs called *beamformers* which, in the transmitter case perform the signal distribution into several channels (4 or 8 in current market ICs) with independent phase and gain control and feed the radiating elements directly or via power amplifiers, depending on the application. In the receiver case the beamformer collects the signals directly from the radiating elements, or through LNAs, and combines them with independent phase and gain control.

The G/T in a phased array system like the one described above is computed in the same manner as their single antenna counterpart. This is possible because the beamformer coherently combines all its channel inputs[3] and so both, the signal and the noise are combined with the same transfer gain, resulting in the same signal-to-noise degradation for all channels, which is the beamformer noise factor.

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[2] $el_x = 0.5$ corresponds to a separation $d/\lambda = 0.5$ between elements aligned in x-direction.

[3] The beamformer channels are programmed to compensate for the phase offsets of the signals detected from a specific direction, thus combining them coherently.

As an example, the effective area of an array of 8x8 radiating elements spaced 7.5mm with an 85% collecting power efficiency is computed as

$$A_{eff,m^2} = 0.85 * \left(8 * \frac{7.5}{10^3}\right)^2 = 3.06 \cdot 10^{-3}\ m^2$$

Or equivalently

$$A_{eff,\ dB} = 10 * \log_{10}\left(A_{eff,m^2}\right) = -25.1\ dB(m^2)$$

This corresponds to a condition of boresight reception and the collected electrical power amounts to

$$P_{ant} = PF * A_{eff,m^2}$$

Which is proportional to the incoming power flux, $PF(W/m^2)$. In the case of dual polarization reception, each radiating element feeds two beamformer channels. A total of 2x64 beamformer channels are used and the input power to each of them is computed as

$$P_{in} = \frac{P_{ant}}{128}$$

This amount of power is delivered by each radiating element at each beamformer input port. It is important to guarantee that this level of power is within the linear operation region of the beamformer input amplifiers, which could be a concern if there were strong interferers within the antenna field-of-view. This is an example of how the dimensioning of an active phased array is equivalent (in terms of antenna gain) not only to the classical passive antenna system, but also to its front-end. Both are intrinsically interrelated in an active array.

All the power P_{ant} collected by the antenna would appear at the output of its beamforming combiner network if all radiating element outputs were added coherently (in phase) with no loss or gain. This is the received power considered when computing the antenna characteristics, and the gain of the aperture is computed as

$$G_a = \frac{4\pi}{\lambda^2} \cdot A_{eff}$$

The actual power delivered by the phased array includes the beamformer coherent gain, which applies when all ports are excited and combined in-phase, plus the gain of the external network combining all coherent beamformer outputs. This is the part that in classical systems is associated to the front-end gain. The array is dimensioned to provide the required gain at its maximum scan angle, accounting for the scan losses, which are modeled as $10 \cdot \log_{10}(Cos(\theta)^p)$.

The equivalent system noise temperature can be computed as

$$T_{sys} \approx T_a + T_0 \cdot \left(L_f F_{BF} - 1\right)$$

Which includes the antenna noise temperature T_a, the beamformer noise factor F_{BF} and feeder losses L_f. The effects of the network combining the different beamformer outputs can be neglected if the beamformer coherent gain is sufficiently high.

Antenna noise temperature T_a is a function of antenna pattern, frequency, and elevation angle from horizon. In the basic approach, sky and atmospheric noise contribute to the antenna noise temperature. Assuming that sky temperature is the dominant part in the current scenario, 25 K are estimated in clear sky and high elevation angles (90º down to 40º), while 50 K could be considered more typical at low elevation angles (40º down to 20º).

Assuming the feeder losses and beamformer noise figure to be 0.5 dB and 2.2 dB, respectively, the resulting system temperature is 24 dBK for clear sky and high elevation conditions, and 24.5 dBK in low elevation angles, which represents a degradation of 0.5 dB.

In the transmitter case, the radiating elements are fed by an active array network of beamformer ICs. Each radiating element being fed in dual polarization by two beamformer channels, providing their signals with the required amplitude and phase differences, and with independent gain and phase control to each radiating element.

Following our previous 8x8 RE example, the power injected into the array is distributed to each radiating element with the required corrections in phase and gain. Once radiated, all this power is combined coherently in a specific direction. The *Equivalent Isotropic Radiated Power* is computed as

$$EIRP = n \cdot P_1 \cdot G_a = P_{in} \cdot G_{distr} \cdot G_a$$

Where $P_1 = P_{in}/n$ is the power that would be delivered to the radiating elements in the case of a lossless distribution ($G_{distr} = 1$). The reference input power for comparison with the classical antennas is $n \cdot P_1$ but, similarly to the receiver case, the antenna gain G_a does not include the extra gain provided by the active distribution network G_{distr}, which is considered part of the front-end.

Both ERIP and G/T increase with antenna gain G_a, which is itself determined by the size of the array aperture, and the number of radiating elements (as the separation between them is defined by design).

The number of radiating elements determines the array power consumption, due to the presence of active beamformer ICs and LNAs or Power Amplifiers (PAs) whose number increases in proportion to the number of radiating elements. This in turn increases the complexity of the array in terms of power consumption, thermal dissipation, and control to configure the different active elements.

The apertures are dimensioned to guarantee a minimum performance in terms of G/T and EIRP at the maximum scan angle, and to keep power consumption and dissipation within appropriate limits for system operation.

Beamforming network

The beamforming network distributes (Tx case) the input signal to the array via several paths independently controlled in gain and phase which are then directly applied to the radiating elements, or through a power amplifier. Fig 7 shows the Tx case. The Rx uses LNAs or direct feed to the different Gain-Phase controlled channels which are subsequently combined into a common output.

The distribution or combining networks can include integrated beamforming ICs (BFICs) together with external passive or active circuitry. The number of required independent paths can be equal to the number of radiating elements, or double (for dual polarisation arrays).

Integrated solutions are required to feed the closely spaced radiating elements in high frequency applications where the wavelength is small compared to the sizes of packaged devices that are assembled on a PCB. The PCB itself is a multilayer structure which can be very difficult to design and manufacture due to the high density of signal, bias and control tracks and the different via interconnections.

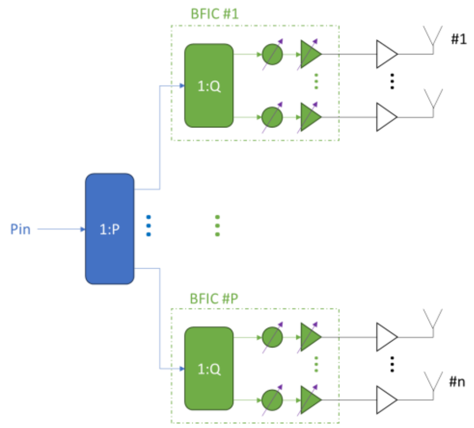

Fig 7. Beamforming network (Tx case)

The working principle of the phased array, as depicted in Fig 3, is to coherently combine (Rx case) the signals detected by the different radiating elements. The signal arriving at an angle with respect to the plane of the array will reach each radiating element with a different delay, which must be compensated by the beamforming network to coherently combine all the radiating element contributions, and in this manner the wanted signal is selected out of those coming from other directions.

The transmitter case is similar. The internally distributed signal reaches each radiating element with a different delay, and therefore the coherent combination of all radiating element contributions only happens in a specific direction.

True time delays can be achieved by passing the signal through sections of lines with different lengths. But this solution occupies space, and it is difficult to implement in current IC technologies. Instead, phase shifters are employed, which are easily integrated. This has the disadvantage of a limitation in the instantaneous operation bandwidth through a phenomenon known as *beam squint*, which is depicted in Fig *8* and Fig *9*.

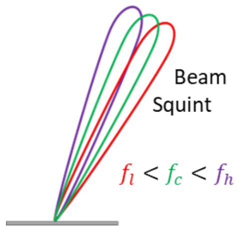

Fig 8. Beam squint effect

The beam squint is the change in beam direction with frequency and it limits the maximum bandwidth of the signals that can be efficiently transmitted or received. Thus, limiting the data throughput, which requires high symbol rates (together with high order modulation schemes).

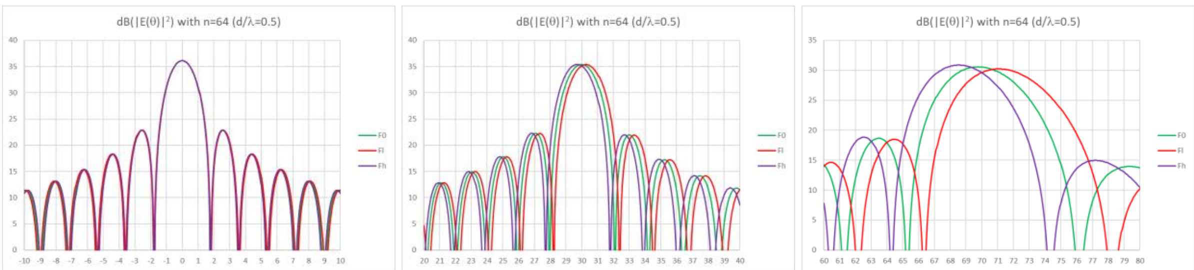

Fig 9. Beam squint degrades with the scan angle

The beam squint effect increases with antenna gain (narrow beams) and scan angle, as depicted in Fig *9*. The derivation of the formulas to compute the beam squint is provided in Appendix II.

Another important parameter that affects the maximum achievable data throughput is linearity. This is more critical in transmission as the distortion provided by the nonlinear power amplification increases with the complexity (order) of the modulation scheme. There is a compromise between linearity and efficiency in power amplifiers, and thermal considerations become critical in solutions with a high density of components.

To keep the component density within reasonable limits (for manufacturing yield, and due to the limited space available) a single PA per radiating element can be used to feed the multibeam signals into each radiating element. But a multibeam is a very complex signal. A single beam contains several channels, with their own independent modulation schemes. The multibeam feed multiplies the number of channels through the PA, where intermodulation distortion takes place. High order modulation schemes are more sensitive to distortion than low order schemes. But they are also the most spectrally efficient[4] and their use is mandatory in high throughput systems.

[4] Spectral efficiency is measured in bit/symbol, and the throughput capacity of a channel [bit/s] is the Symbol rate [symbol/s] multiplied by the spectral efficiency.

CONCLUSION

The principal use of Electronic Beam Steering Arrays for Satellite Communication is to establish efficient high data rate radio links with multiple moving satellites.

The availability of multiple independent beams per array is an essential quality that offers these systems a competitive advantage with respect to their high performing classical counterparts, the well-known dish antennas with mechanical tracking capability.

Among the most relevant parameters for a radio link are EIRP in transmission, and G/T in reception, for a given channel bandwidth and modulation scheme, which subsequently determines the maximum available data rate. Other parameters such as the quality of polarization affect the link losses. Rejection of secondary radiating lobes is also required for system compliance with radio regulations, imposing a radiation mask to limit unwanted emissions or mitigate the systems' vulnerability to interference from external sources.

Compliance to radiation mask is achieved in part by setting the distance between radiation elements to a fraction of the wavelength, which reduces for higher scan angle requirements. This imposes a high density of integration for systems operating a very high frequency (sub-cm wavelengths).

The requirement for multibeam operation further intensifies the density of electronic components, with the increased complexity on the bias, control and thermal dissipation subsystems.

Quality of polarization is achieved by the design and feeding of the radiating elements. Patch antennas are a common type of radiating element.

The directivity (beamwidth) requirement imposes a minimum array size and this in turn affects the EIRP and G/T performance.

EIRP depends on the antenna Gain (directivity degraded by feeder loss) and the total amount of power into the radiating elements (single power per element multiplied by their number). It is possible to act on the EIRP by changing the power fed into the radiating elements.

G/T depends on the antenna Gain and, on the antenna and system noise temperatures. Antenna noise temperature is an external parameter which is affected by the beam direction. System noise temperature is an internal parameter which mostly depends on the first stage (Low Noise Amplifier) performance, provided that its Gain is sufficiently high to neglect the other system contributions to noise. As the system internal noise is minimized by design, the only deign parameter left to optimize G/T is the array size.

Multiple panel arrays with different fixed orientations can be operated together to provide hemispherical coverage, as in CelestiaUK's eScan Gateway for Mega-constellations (*Fig 10*), which can be configured with up to 7 panels, having a capacity of four beams per panel in two polarizations per beam, and a simultaneous operational bandwidth of 2.5 GHz per beam and polarization. This provides a system capacity of 140 GHz (each direction) for data links with 28 different satellites simultaneously.

Handover operations are performed between panels to provide link continuity during satellite passes.

Fig 10. CelestiaUK eScan Gateway for Mega-constellations

APPENDIX I: Combined Electrical Field from an Array of Radiating Elements

The addition of the electric fields from n radiating elements, at a specific point in the space, is a linear process and can be expressed as

$$E(r,\theta,\varphi) = \sum_{k=0}^{n-1} E_k(r_k,\theta,\varphi)$$

Where the direction is expressed by the angles azimuth φ, and polar θ in a spherical coordinate system, and

$$E_k(r_k,\theta,\varphi) = E_{0,k}(r_0,\theta,\varphi) \cdot \left(\frac{r_0}{r_k}\right) \cdot e^{j2\pi \cdot \frac{r_k - r_0}{\lambda}} \cdot \sqrt{G_k} \cdot e^{j\phi_k}$$

Is the field contributed by each element at a distance r_k, where lays a plane perpendicular to the radiating direction at a distance r from a reference point of the array, known as its *centre of phase*. This term contains four main components:

- $E_{0,k}(r_0,\theta,\varphi)$ is the electrical field produced by the kth element at a reference distance r_0 from its own *centre of phase*. This term accounts for the directivity of the radiating element.
- $\left(\frac{r_0}{r_k}\right)$ this term can be approximated by $\left(\frac{r_0}{r}\right)$ in the far field, where $r \gg \lambda$
- $e^{j2\pi \cdot \frac{r_k - r_0}{\lambda}}$ accounts for the electrical path to the plane.
- $\sqrt{G_k} \cdot e^{j\phi_k}$ accounts for the gain and phase shift effected by the beamforming network to the electrical signal which is common to all radiating elements.

The combined electrical field expression admits an analytical form in some specific cases:

A) Identical n-elements lying on a line with constant separation d between them.

In the direction of the line of radiating elements, at a reference distance from the array, the relative individual field contributions can be expressed as

$$E_k(\theta) = E_0 \cdot \sqrt{D(\theta)} \cdot e^{j2\pi \cdot \frac{k\ell}{\lambda}} \cdot e^{j\phi_k}$$

- The function $D(\theta) = \cos(\theta)^p$ is a close match to the observed performance, with p in the range 1.2-1.4.
- $e^{j2\pi\cdot\frac{k\ell}{\lambda}}$ accounts for the accumulated path length difference between radiating elements, for fields radiated in the θ direction, with associated phase shift $2\pi\ell/\lambda$ and $\ell = d \cdot \sin(\theta)$.
- $e^{j\emptyset_k}$ is the phase shift generated by the beamforming network, with same gain applied to all radiating elements, and $\emptyset_k = -2\pi k \cdot \left(\frac{d}{\lambda_0}\right) \cdot \sin(\theta_0)$ at the frequency c/λ_0.

It is straightforward to see that the boresight contributions (θ = 0) do not suffer any difference in path length, and the resulting electrical field is $E(0) = n \cdot E_0$.

The general expression of the electrical field is

$$E(\theta) = E_0 \cdot \sqrt{D(\theta)} \cdot \sum_{k=0}^{n-1} e^{j2\pi k\cdot\left(\frac{d}{\lambda}\cdot\sin(\theta)-\frac{d}{\lambda_0}\cdot\sin(\theta_0)\right)}$$

Which admits the closed form

$$E(\theta) = n \cdot E_0 \cdot \sqrt{D(\theta)} \cdot e^{j(n-1)\frac{x}{2}} \cdot \frac{sinc\left(n\frac{x}{2}\right)}{sinc\left(\frac{x}{2}\right)}$$

With

$$x = 2\pi \cdot \left(\frac{d}{\lambda} \cdot \sin(\theta) - \frac{d}{\lambda_0} \cdot \sin(\theta_0)\right)$$

The derivation is as follows:

$$\sum_{k=0}^{n-1} e^{jkx} = 1 + \sum_{k=1}^{n-1} e^{jkx} = 1 + e^{jx} \cdot \sum_{k=0}^{n-2} e^{jkx} = 1 + e^{jx} \cdot \sum_{k=0}^{n-1} e^{jkx} - e^{jnx}$$

Hence

$$\sum_{k=0}^{n-1} e^{jkx} = \frac{1-e^{jnx}}{1-e^{jx}} = e^{j(n-1)\frac{x}{2}} \cdot \frac{\sin\left(n\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} = e^{j(n-1)\frac{x}{2}} \cdot n \cdot \frac{\mathrm{sinc}\left(n\frac{x}{2}\right)}{\mathrm{sinc}\left(\frac{x}{2}\right)}$$

B) Identical n x m elements lying on a plane grid with constant separation d between them

The general expression of the electrical field is

$$E(\theta,\varphi) = E_0 \cdot \sqrt{D(\theta)} \cdot \sum_{k=0}^{n-1} e^{j2\pi k\cdot\left(\frac{d}{\lambda}\cdot\sin(\alpha)-\frac{d}{\lambda_0}\cdot\sin(\alpha_0)\right)} \cdot \sum_{l=0}^{m-1} e^{j2\pi l\cdot\left(\frac{d}{\lambda}\cdot\sin(\beta)-\frac{d}{\lambda_0}\cdot\sin(\beta_0)\right)}$$

With

$$\tan(\alpha) = tan(\theta) \cdot cos(\varphi)$$

$$\tan(\beta) = tan(\theta) \cdot sin(\varphi)$$

The resulting field admits the closed form

$$E(\theta) = n \cdot m \cdot E_0 \cdot \sqrt{D(\theta)} \cdot e^{j(n-1)\frac{x}{2}} \cdot \frac{sinc\left(n\frac{x}{2}\right)}{sinc\left(\frac{x}{2}\right)} \cdot e^{j(m-1)\frac{y}{2}} \cdot \frac{sinc\left(m\frac{y}{2}\right)}{sinc\left(\frac{y}{2}\right)}$$

With

$$x = 2\pi \cdot \left(\frac{d}{\lambda} \cdot \sin(\alpha) - \frac{d}{\lambda_0} \cdot \sin(\alpha_0)\right)$$

$$y = 2\pi \cdot \left(\frac{d}{\lambda} \cdot \sin(\beta) - \frac{d}{\lambda_0} \cdot \sin(\beta_0)\right)$$

APPENDIX II: Beam Squint

The beam squint effect is produced when a phase correction independent of frequency is produced by the internal beamforming network.

In the case A of Appendix I (case B is similar), the beam has its maximum amplitude at the angle θ_{peak} satisfying $x = 0$, thus

$$\frac{\sin(\theta_{peak})}{\lambda} = \frac{\sin(\theta_0)}{\lambda_0}$$

Hence

$$\theta_{peak} = arcsin\left(\frac{\lambda}{\lambda_0} \cdot sin(\theta_0)\right) = arcsin\left(\frac{f_0}{f} \cdot sin(\theta_0)\right)$$

And this implies that $\theta_{peak} \neq \theta_0$ when $f \neq f_0$, producing a pointing error

$$\Delta\theta(f) = arcsin\left(\frac{f_0}{f} \cdot sin(\theta_0)\right) - \theta_0$$

To overcome this effect the beamforming network must operate a true time delay with

$$\theta_k = -2\pi k \cdot \left(\frac{d}{\lambda}\right) \cdot \sin(\theta_0) = -\frac{2\pi f}{c} \cdot kd \sin(\theta_0)$$

The term

$$-\frac{2\pi}{c} \cdot kd \sin(\theta_0)$$

is referred to as the *phase slope*.