The Effects of Pole Locations on the Transfer Function of a Symmetrical Four Section Filter

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A study of pole locations in four section filters begins with an explanation of the possible cross coupling configurations. Then the theory behind the synchronously tuned filter is derived and examples are given and studied. The input impedance is derived, and the circuit element values are calculated. User examples can be entered and analysed to further acquaint the operator with the effects of the pole locations on the filter's insertion loss, return loss, group delay and element value spread. The excel file in the proceedings allows the user to adjust sliders to alter all the filter parameters and observe the results.

1. INTRODUCTION

A generalised four section filter would have five couplings from each node. One to each of the other nodes, figure 1.1 show a simplification of this general case. Any synchronously tuned filter – the coefficients of S21 are not complex – can be realised by the network shown in figure 1.2 [1]. Any generalised filter can be realised by this network with the addition of diagonal cross couplings as in figure 1, but this is not necessarily the desired implementation. Extracted poles may be desired, independent control of pole frequencies may be needed or physical constraints could dictate a particular realisation. The relationship of the cross-coupling signs and structures to the pole locations can be established. The bandpass theory is not examined in this paper because it requires root finding of degree 8 functions, and that is painful in Excel, this work is constrained to polynomials that can be rooted by the quartic formula.

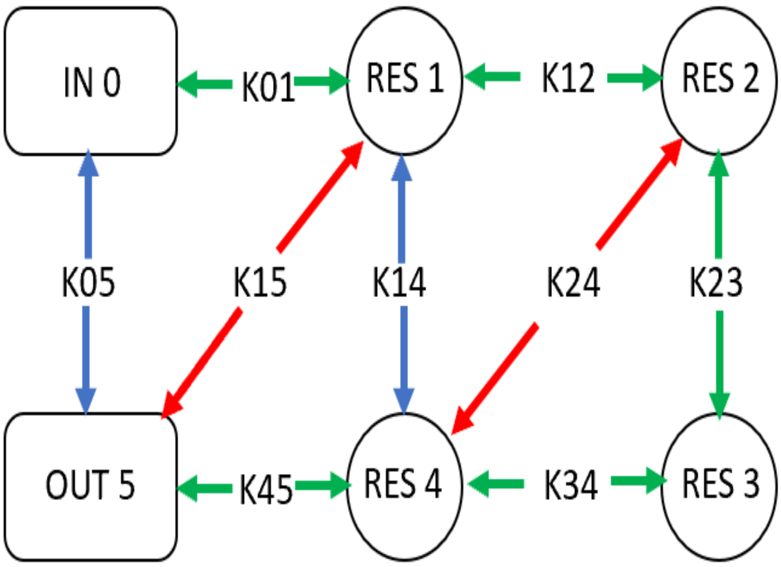


Figure 1.1. Generalised four section filter

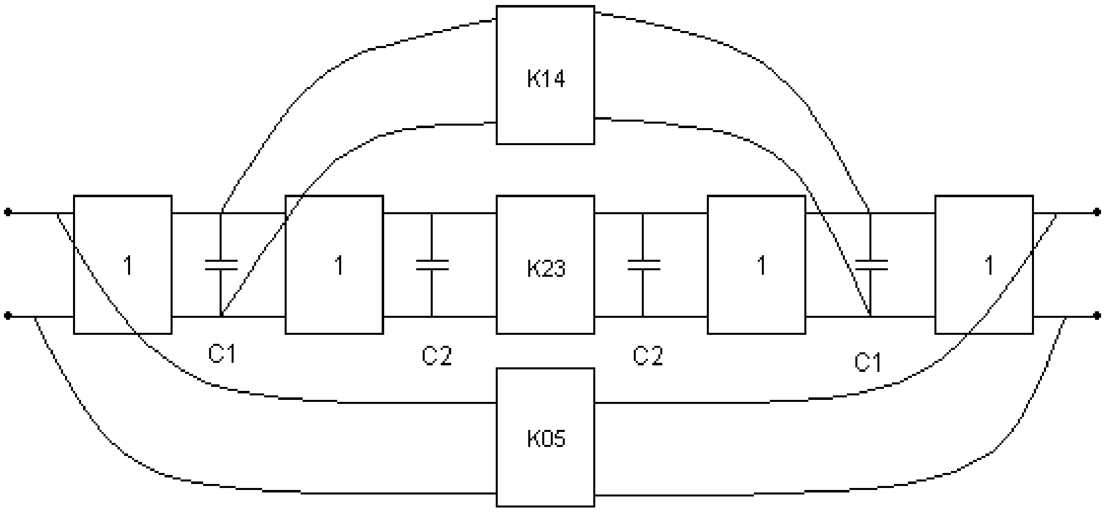


Figure 1.2 Symmetrical four section low-pass Prototype Network

In the synchronously tuned case (K_{24}=K_{15}=0), there are three main possibilities for cross coupling. If all the main line (green) couplings are considered positive, then k14 and k05 can be positive or negative. If the signs of k_{23}, k_{14} and k_{05} are all positive then a quadrupole is produced (self-equalisation); if they alternate in sign, then four transmission zeroes are produced ($\pm j\omega_1$ and $\pm j\omega_2$). In the third case, i.e. Where two of the inverters have the same sign and the end one has the opposite sign then two transmission zeroes are produced along with a pair of real axis zeroes which can equalise the response.

2. FORMATION OF INPUT ADMITTANCE

To study the symmetrical four-section filter (figure 1.2), the input impedance must be generated and then equations must be derived to calculate the circuit element values from this input impedance**.** In general:

$$|S_{12}(j\omega)|^2 = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

Factors for an equiripple bandpass characteristic function, $C_n(\omega)$ have been established [2,3,4]. In the general case with a passband that extends from K to 1/K (K<1) these factors can be written in the following form:

$$\frac{A + B\omega^2 + C\sqrt{(\omega^2-k^2)(\omega^2-1/k^2)}}{(1-k^4)(\omega^2-\omega_i^2)} \quad \ldots 2.2$$

$$A = 2k^2 - \omega_i^2 k^4 - \omega_i^2 \qquad B = 2\omega_i^2 k^2 - k^4 - 1 \qquad C = \pm 2k^2\sqrt{(\omega_i^2-k^2)(\omega_i^2-1/k^2)}$$

The characteristic function is formed by taking the rational part of the product of these factors.

This general bandpass factor places a transmission zero at $\pm\omega_i$, where ω_i can be complex. For a lowpass (symmetrical) function normalised to a cutoff frequency of 1 rad/S, a factor:

$$\omega + \sqrt{\omega^2 - 1}$$

will place a zero at infinite frequency, and

$$\left(\left[\frac{(2\omega_i^2 - 1)\omega^2 - \omega_1^2 + 2\omega\omega_1\sqrt{\omega_1^2 - 1}\sqrt{\omega^2 - 1}}{\omega_1^2 - \omega^2}\right]\right) \quad \ldots\ 2.3$$

will place a zero at $\pm\omega_i$.

Taking the rational part of the product of the factors in 2.2 forms the characteristic function. Hence, for a generalised symmetrical four section lowpass characteristic function with poles located at $\pm j\omega_1$ and $\pm j\omega_2$

$$C_n(\omega) = rational$$

$$\left(\left[\frac{(2\omega_1^2 - 1)\omega^2 - \omega_1^2 + 2\omega\omega_1\sqrt{\omega_1^2 - 1}\sqrt{\omega^2 - 1}}{\omega_1^2 - \omega^2}\right]\left[\frac{(2\omega_2^2 - 1)\omega^2 - \omega_2^2 + 2\omega\omega_2\sqrt{\omega_2^2 - 1}\sqrt{\omega^2 - 1}}{\omega_2^2 - \omega^2}\right]\right)$$

let:

$$C_n(\omega) = \frac{A\omega^4 + B\omega^2 + C}{\omega^4 - (\omega_1^2 + \omega_2^2)\omega^2 + \omega_1^2\omega_2^2} = \frac{C_n num(\omega)}{C_n den(\omega)} \quad \ldots\ 2.4$$

Then:

$$A = 4\omega_1^2\omega_2^2 - 2\omega_1^2 - 2\omega_2^2 + 1 + 4\omega_1\omega_2\sqrt{\omega_1^2 - 1}\sqrt{\omega_2^2 - 1}$$

$$B = -\omega_1^2(2\omega_2^2 - 1) - \omega_2^2(2\omega_1^2 - 1) - 4\omega_1\omega_2\sqrt{\omega_1^2 - 1}\sqrt{\omega_2^2 - 1} \qquad C = \omega_1^2\omega_2^2$$

Start by forming the characteristic function of an example lowpass filter:

Cut off frequency: 1000MHz Return loss (rl): 20.0dB

Pole location one: 0 + 2100j Pole location two: 0 + 2800j

Impedance: 1Ω

Calculate:

ε = sqrt(1/(10$^{(rl/10)}$-1))= 0.10050 ripple (dB) = 10.log$_{10}$(1+ε^2) = 0.0436

Therefore, from equations 2.3 and 2.4 the characteristic function is given by:

$$C_n(\omega) = \frac{7.82\omega^2 - 4.41 + 7.76\omega\sqrt{\omega^2 - 1}}{4.41 - \omega^2} \cdot \frac{14.6459\omega^2 - 7.84 + 14.646\omega\sqrt{\omega^2 - 1}}{7.84 - \omega^2}$$

$$= \frac{114.8\omega^4 - 126.05\omega^2 - 34.5744 + 113.59\ \omega^2\ (\omega^2 - 1\ )}{\omega^4 - 12.25\omega^2 + 34.574}$$

$$\frac{228.388\omega^4 - 239.638\omega^2 + 34.574}{\omega^4 - 12.25\omega^2 + 34.574} = \frac{N(\omega)}{D(\omega)}$$

$$\lfloor S_{11}(\omega) \rfloor^2 = \frac{\varepsilon^2 N^2(\omega)}{D^2(\omega) + \varepsilon^2 N^2(\omega)}$$

... 2.5

$$\lfloor S_{12}(\omega) \rfloor^2 = \frac{D^2(w)}{D^2(\omega) + \varepsilon^2 N^2(\omega)}$$

Therefore, the denominator is:

$$den \left\lfloor S_{ij}(j\omega) \right\rfloor^2 = 527.88\omega^8 - 1130.688\omega^6 + 958.8\omega^4 - 1014.4536\omega^2 + 1207.4638$$

$$= \omega^8 - 2.141\omega^6 + 1.8163\omega^4 - 1.9217\omega^2 + 2.2874$$

The points of perfect transmission can now be calculated. These occur when S11 is zero, i.e. they are the zeros of N(ω).

Using the quadratic equation the zeros of N(ω) in ω^2 are 0.8766 and 0.1727. And the roots in ω are therefore 0.9362 and 0.4156, and these are the normalised points of perfect transmission.

Next, we find the factors of the denominator polynomial using the quartic equation. The roots in ω^2 are $0.2543 \pm j1.026336$ and $1.32478 + j0.5393$. The roots in ω^2 are $0.63367 \pm j0.80993$ and $1.1737 + j0.22974$. Thus, the roots in p (jω) are $0.80993 \pm j0.63367$ and $0.22974 + j1.1737$. Next multiply out the left half plane roots in p.

$$denS_{12}(p) = denS_{11}(p) = 22.976(p^4 + 2.0796p^3 + 3.2326p^2 + 2.8032p + 1.5127)$$

The multiplier is the square root of the leading coefficient of the denominator squared function. In this example 527.88.

$$denS_{ij}(p) = 22.976p^4 + 47.77936p^3 + 74.2701p^2 + 64.405p + 34.755$$

From equation 2.5:

$$numS_{11}(p) = 22.958p^4 + 24.0845p^2 + 3.4755$$

So, for analysis purposes we can form:

$$S_{11}(j\omega) = \frac{22.958\omega^4 - 24.0845\omega^2 + 3.4755}{22.976\omega^4 - 47.77936\omega^3 - 74.2701\omega^2 + 64.405\omega + 34.755}$$

And:

$$S_{12}(j\omega) = \frac{\omega^4 - 12.25\omega^2 + 34.5744}{22.976\omega^4 - 47.77936\omega^3 - 74.2701\omega^2 + 64.405\omega + 34.755}$$

Since:

$$Z_{in}(p) = \frac{1 + S_{11}(p)}{1 - S_{11}(p)}$$

Then:

$$Z_{in}(p) = \frac{45.929p^4 + 47.77936p^3 + 98.3546p^2 + 64.405p + 38.2299}{0.02177p^4 + 47.77936p^3 + 50.1855p^2 + 64.405p + 31.2802}$$

3. CALCULATION OF ELEMENT VALUES

As derived in the spreadsheet 'Input Z' tab, the input impedance of figure 1.2 is:

$$\frac{K_{05}^2 C_1^2 C_2^2 K_{14}^2 K_{23}^2 P^4 + K_{05}^2 C_1 C_2^2 K_{14}^2 K_{23}^2 P^3 + K_{05}^2 (C_1^2 K_{14}^2 + 2C_1 C_2 K_{14}^2 K_{23}^2 + C_2^2 K_{23}^2) P^2 + K_{05}^2 K_{14}^2 (C_2 K_{23}^2 + C_1) P + K_{05}^2 (1 - K_{23} K_{14})^2}{C_1^2 C_2^2 K_{14}^2 K_{23}^2 P^4 + K_{05}^2 C_1 C_2^2 K_{14}^2 K_{23}^2 P^3 + (2C_2^2 K_{23}^2 K_{05} K_{14} - 2C_1 C_2 K_{14}^2 K_{23}^2 - K_{05}^2 K_{14}^2 C_2^2 K_{23}^2 - C_2^2 K_{23}^2 - C_1^2 K_{14}^2) P^2 + K_{05}^2 K_{14}^2 (C_2 K_{23}^2 + C_1) P + (K_{14} K_{23} + K_{05} K_{14} - 1)^2}$$

If:

$$Z_{in}(P) = \frac{Z_{num4}P^4 + Z_{num3}P^3 + Z_{num2}P^2 + Z_{num1}P + Z_{num0}}{Z_{den4}P^4 + Z_{den3}P^3 + Z_{den2}P^2 + Z_{den1}P + Z_{den0}}$$

Then by inspection:

$$K_{05}^2 = \frac{Z_{num4}}{Z_{den4}}$$

And:

$$C_1 = \frac{Z_{num4}}{Z_{num3}}$$

Since C_1 and K_{05} are known, taking the ratio of the appropriate terms of the input impedance forms three equations in C_2, K_{14}, and K_{23}:

$$\sqrt{\frac{Z_{den0}}{Z_{den4}}} = \frac{1}{C_1 C_2}\left(1 + \frac{K_{05}}{K_{23}} - \frac{1}{K_{14}K_{23}}\right) \qquad \frac{Z_{num1}}{Z_{num3}} = \frac{1}{C_2^2 K_{23}^2} + \frac{1}{C_1 C_2}$$

$$\sqrt{\frac{Z_{num0}}{Z_{num4}}} = \frac{1}{C_1 C_2}\left(1 - \frac{1}{K_{14}K_{23}}\right)$$

Substituting the last of these into the first and rearranging gives:

so that the product C_2K_{23} can be calculated. Rearranging the middle equation of the three to calculate C_2:

$$C_2 K_{23} = \frac{K_{05}}{C_1\left(\sqrt{\frac{Z_{den0}}{Z_{den4}}} - \sqrt{\frac{Z_{num0}}{Z_{num4}}}\right)} \qquad \frac{1}{C_2} = C_1\left(\frac{Z_{num1}}{Z_{num3}} - \frac{1}{C_2^2 K_{23}^2}\right)$$

K_{23} can now be calculated from C_2K_{23}. Finally rearrange the third equation to give K_{14}

$$\frac{1}{K_{14}} = K_{23}\left(1 - C_1C_2\sqrt{\frac{Z_{numo}}{Z_{num4}}}\right)$$

Once the lowpass values have been calculated, the capacitors can be resonated to form a bandpass filter. If the passband of the filter goes from frequency k to 1/k, then a bandwidth scaling factor (α) should be used, where:

$$\alpha = \frac{K}{1 - K^2}$$

4. EVALUATION OF DIFFERENT POLE DISTRIBUTIONS

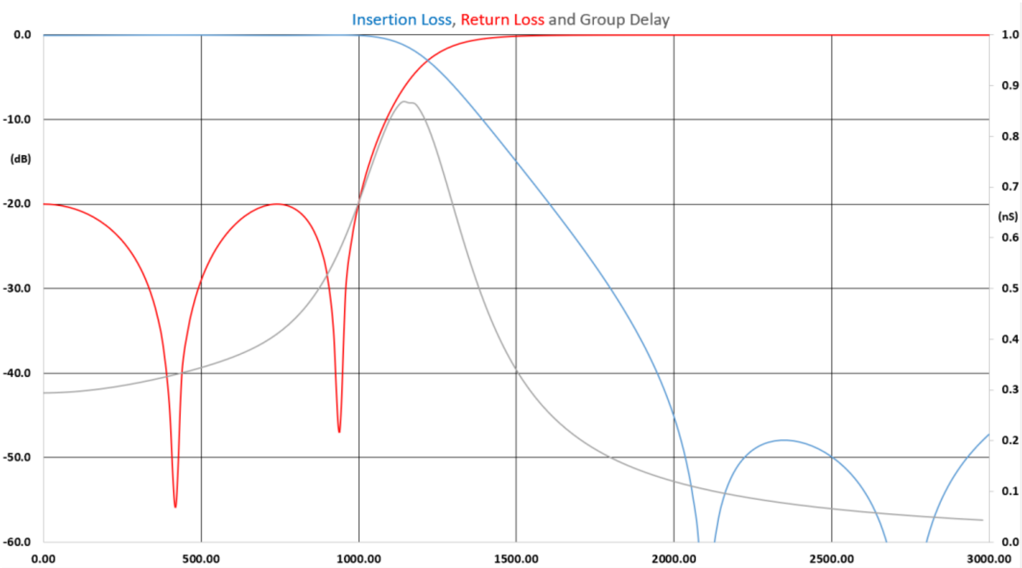
4.1 Two purely imaginary (real frequency) zeros

Cutoff freq	Return Loss	Pole Location σ	Pole location jω
(MHz)	(dB)		frequency MHz))
1000.000	20.000	0.000	2102.000
Q factor (all Ls&Cs)		0.000	2740.000
10000000	Use the sliders below to enter these values		

1. two real frequency poles

	σ	jω
1	0.00	2102.00
	0.00	-2102.00
	0.00	2740.00
	0.00	-2740.00

c1= 0.961913853
c2= 1.390609115
k23= 0.928040967
k14= -4.853587139
k05= 43.94157366



Most commonly used, most selective

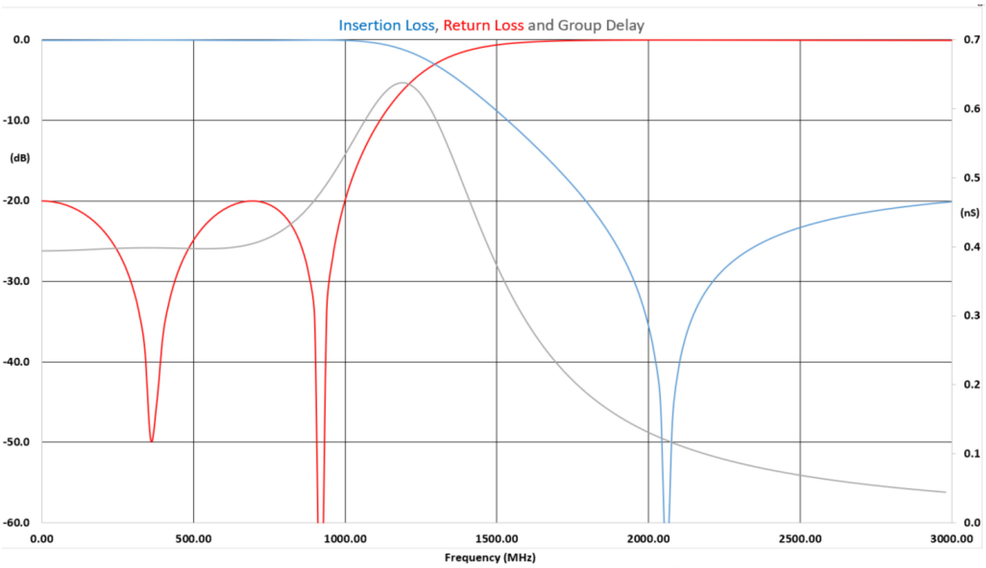
Group delay variation increases as zeros go down towards the passband, band edge loss increases (proportional to group delay)

4.2 One purely imaginary (real frequency) and one purely real zero

Cutoff freq (MHz)	Return Loss (dB)	Pole Location σ	Pole location jω frequency MHz))
1000.000	20.000	0.000	2062.000
Q factor (all Ls&Cs)		1390.000	0.000
10000000	Use the sliders below to enter these values		

2. One real axis (equalising) pole and one actual frequency pole		
2	0.00	2062.00
	0.00	-2062.00
	1390.00	0.00
	-1390.00	0.00

c1= 0.004453379
c2= 239.5272294
k23= -0.006726904
k14= 14.7467452
k05= 0.068863736



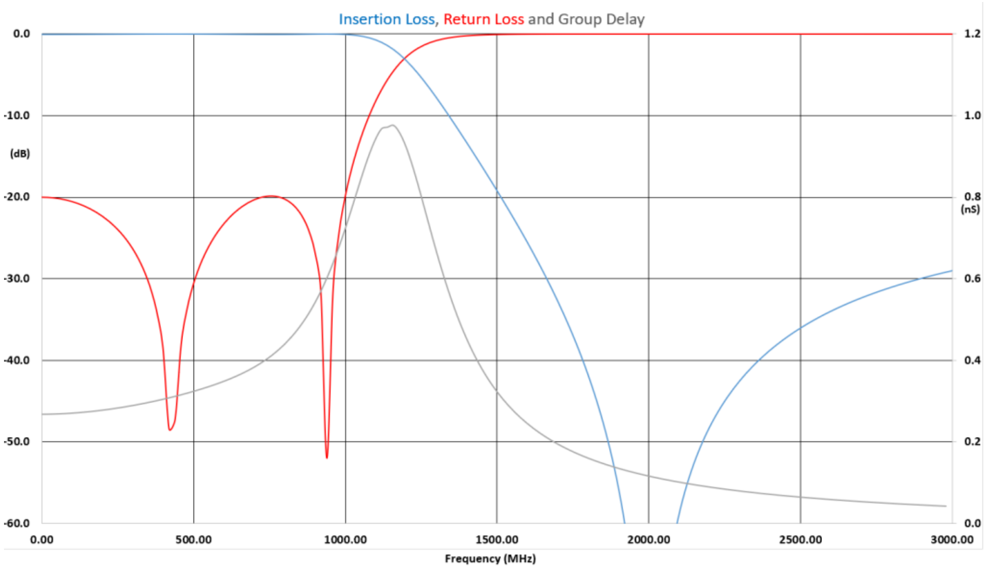
4.3 A quadrupole

Cutoff freq (MHz)	Return Loss (dB)	Pole Location σ	Pole location jω frequency MHz))
1000.000	20.000	1445.000	2468.000
Q factor (all Ls&Cs)		-1445.000	2468.000
10000	Use the sliders below to enter these values		

3. A quadrupole		
3	1445.00	2468.00
	-1445.00	-2468.00
	-1445.00	2468.00
	1445.00	-2468.00

c1= 0.979548905
c2= 1.423855079
k23= 0.886367855
k14= -3.512134258
k05= 19.74422306

Zero frequency $\sqrt{(Im^2 - Re^2)} = \sqrt{2468^2 - 1445^2}$ **= 2000.75MHz**

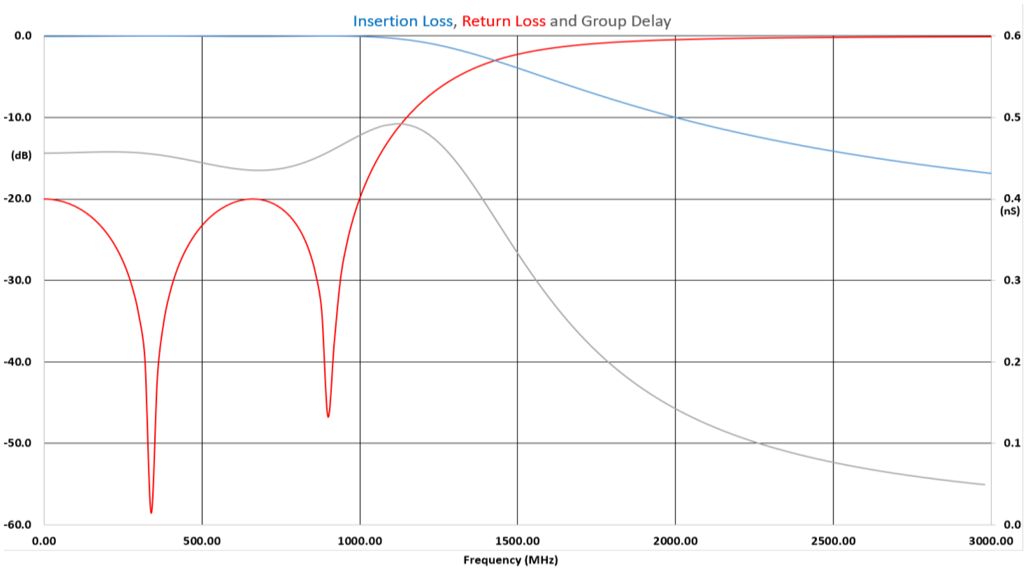


4.4 Two purely real zeros

Cutoff freq (MHz)	Return Loss (dB)	Pole Location σ	Pole location jω frequency MHz))
1000.000	20.000	1445.000	0.000
Q factor (all Ls&Cs)		2898.000	0.000
10000	Use the sliders below to enter these values		

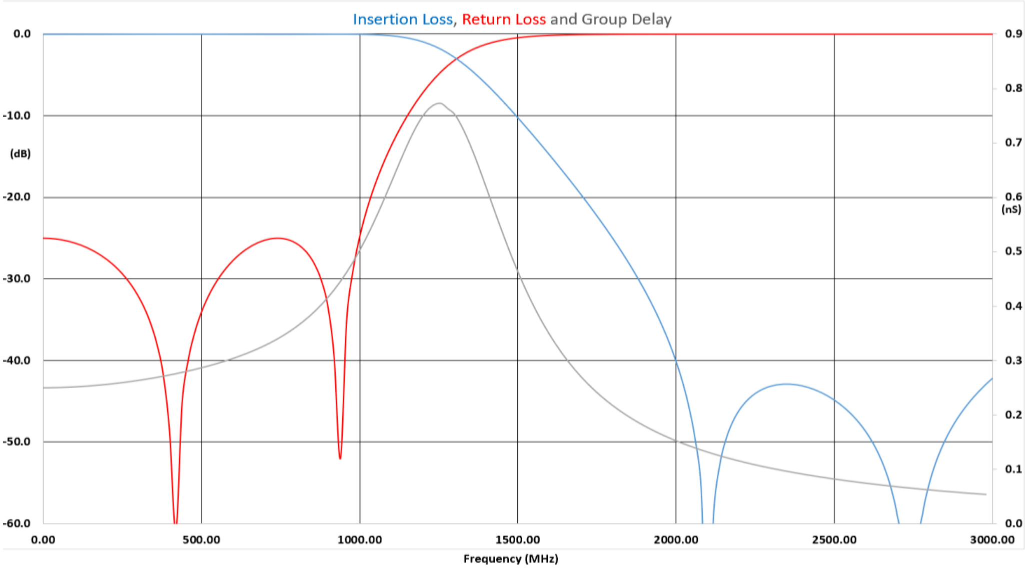
4. Two real axis poles for maximum delay flatness		
4	1445.00	0.00
	-1445.00	0.00
	2898.00	0.00
	-2898.00	0.00

c1= 0.881347712
c2= 1.262712561
k23= 1.408697985
k14= 3.719742384
k05= 36.68220524

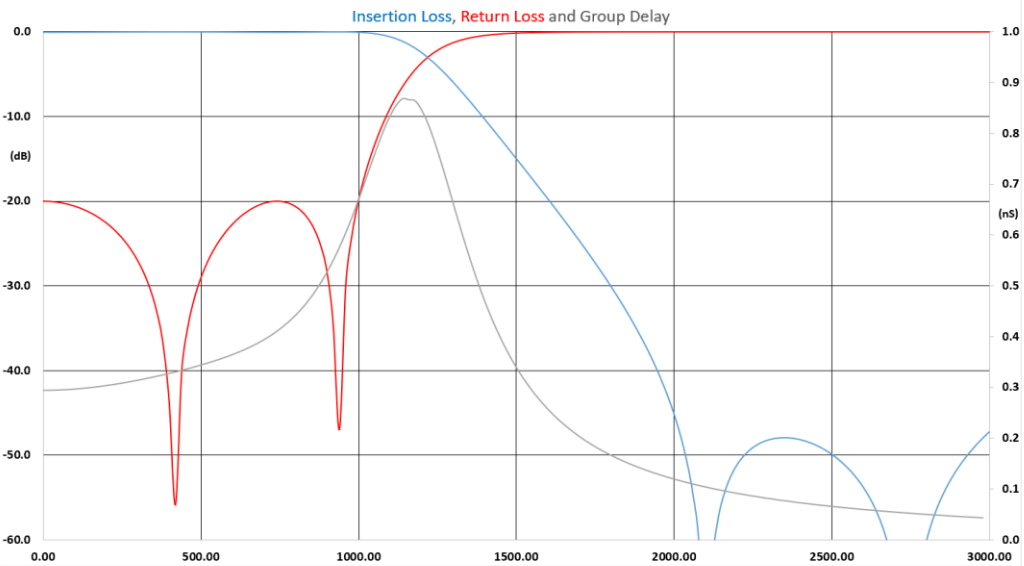


Used where delay flatness (linear phase) response is paramount.

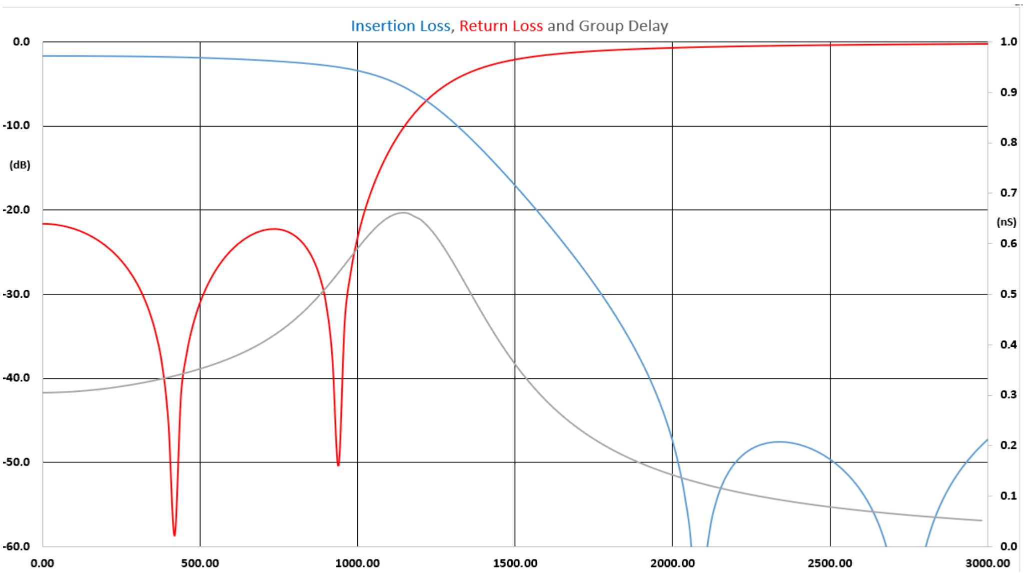
Effects of return loss and finite Q



Return loss = 25dB



Return Loss = 20dB, Q = 10,000



Return loss = 20dB, Q = 20

5. Conclusions

A spreadsheet has been produced that allows the user to investigate the effects of changing pole locations (and other parameters) on the characteristics of a finite loss filter.

6. References

[1] J. David Rhodes: "Theory of Electrical Filters", Wiley – Interscience, 1976

[2] P.D. Sleigh: “Asymmetric Filter Design for Satellite Communication Applications”, IEE Colloquium on Microwave Filters, IEE Colloquium Digest, 1982/4, Jan 1982

[3] R.J. Cameron: “Fast generation of Chebyshev Filter Prototypes with Asymmetrically – Prescribed Transmission Zeros”, ESA Journal 1982 Vol. 6

[4] D.S.G. Chambers: "Design, analysis and realisation of microwave filters" PhD University of Leeds 1988

[5] J. David Rhodes: "Theory of Electrical Filters", Wiley – Interscience, 1976. pp 68-75